A numerical exploration of the dynamical behaviour of q-deformed nonlinear maps

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Abstract: In this paper we explore the dynamical behaviour of the q-deformed versions of widely studied 1D nonlinear map—the Gaussian map and another famous 2D nonlinear map—the Henon map. The Gaussian map is perhaps the only 1D nonlinear map which exhibits the co-existing attractors. In this study we particularly compare the dynamical behaviour of the Gaussian map and q-deformed Gaussian map with a special attention on the regions of the parameter space, where these maps exhibit co-existing attractors. We also generalize the q-deformation scheme of 1D nonlinear map to the 2D case and apply it to the widely studied 2D quadratic map—the Henon map which is the simplest nonlinear model exhibiting strange attractor.

Keywords: q-deformation, Gaussian map, Henon map, Lyapunov exponent, Chaos, co-existing attractors

1. Introduction
The fascinating theory of quantum groups has attracted considerable interest of physicists and mathematicians towards the special branch of mathematics dealing with q-deformed versions of numbers, series, functions, exponentials, differentials etc. (i.e. the q-mathematics) [1]. The q-deformation of any function is to introduce an additional parameter (q) in the definition of function in such a way that under the limit \( q \to 1 \), the original function is recovered. Hence there exist several deformations of the same function. A recent study [2] induced the study of q-deformation of nonlinear dynamical system, where a scheme for the q-deformation of nonlinear maps (in analogy to the q-deformation of numbers, functions, series etc.) has been suggested. In this study authors have shown that the q-deformed version of logistic map (q-logistic map) exhibits a variety of interesting dynamical behaviours (which also exist in the canonical logistic map) including the co-existing attractors, which are not present in canonical logistic map. Further Patidar [3] and Patidar et al. [4] analyzed the dynamical behaviour of q-deformed version of another famous 1D map—the Gaussian map, which is perhaps the only known 1D map, exhibiting co-existing attractors.

In this paper, we present the results of our recent analysis of the dynamical behaviour of various q-deformed maps. Particularly, we report the results of numerical exploration of the dynamical behaviour of q-deformed versions of widely studied 1D nonlinear map—the Gaussian map and another famous 2D nonlinear map—the Henon map.
2. The q-deformation of nonlinear maps

In this section, we briefly introduce the q-deformation scheme for the nonlinear maps proposed in [2], however for a thorough discussion on q-deformation, we refer the readers to [2] and references cited therein. In general a q-exponential function is given by

\[ [e^x]_q = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \]  

where \([n]_q! = [1]_q[n]_q[2]_q[3]_q \ldots [n-1]_q[n]_q\), \([n]_q = (1 - q^n)/(1 - q)\) and the subscript outside the square bracket denotes the q-deformed version of the argument inside the square bracket. It is clear from the above expressions that under the limit \(q \to 1\), \([n]_q \to n\), \([n]_q! \to n!\) and hence \([e^x]_q \to e^x\).

Another definition of q-exponential function proposed by Borges [5] is

\[ [e^x]_q = 1 + \sum_{n=1}^{\infty} \frac{Q_{n-1} x^n}{n!}, \]  

where \(Q_n = l(q)(2q - l) \ldots (nq - (n - 1))\).

With the substitution \(l - q = \varepsilon\), Eq. (2) becomes

\[ [e^x]_\varepsilon = \sum_{n=0}^{\infty} \frac{T_n x^n}{n!}, \]  

where \(T_n = l^{(n-1)}\) (for \(n = 0\)) and \(T_n = l(1 - \varepsilon)(1 - 2\varepsilon) \ldots (1 - (n-1)\varepsilon)\) (for \(n \geq 1\)). It is clear that under the limit \(q \to 1\) i.e. \(\varepsilon \to 0\), \([e^x]_\varepsilon \to e^x\).

If we compare Eqs. (1) and (3), we obtain a new definition for the deformation of numbers as

\[ [n]_\varepsilon = \frac{n}{1 - (n - 1)\varepsilon}, \]

Under the limit \(\varepsilon \to 0\), \([n]_\varepsilon \to n\).

If we extend the above definition of deformation of numbers to any real number \(x\) as

\[ [x]_\varepsilon = \frac{x}{1 - \varepsilon(x - 1)}, \]

such that

\[ \lim_{\varepsilon \to 0} [x]_\varepsilon = x. \]  

Equivalently, Eq. (5) can be written as
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\[ [x]_q = \frac{x}{1 - (1 - q)(x - 1)}, \]  

(6)
such that

\[ \lim_{q \to 1} [x]_q = x. \]  

The q-deformation scheme for 1D nonlinear map [2] of the form \( x_{n+1} = f(x_n) \) is given by \( x_{n+1} = f([x_n]_q) \).

In this paper we are aimed to numerically analyze the dynamical behaviour of the q-deformed versions of the widely studied 1D nonlinear map—the Gaussian map and another famous 2D nonlinear map—the Henon map. In the next subsections, we briefly introduce the Gaussian and Henon maps and their q-deformed versions.

2.1. The q-deformed Gaussian map

The Gaussian map [6] is based on the Gaussian exponential function. It is characterized by two parameters \( b \) and \( c \) as follows:

\[ x_{n+1} = e^{-bx_n^2} + c \quad \text{(Gaussian map)} \]  

(7)
The q-deformed version (i.e. \( q \)-Gaussian map) of the Gaussian map by following the above definition of the q-deformation of 1D nonlinear map is given as:

\[ x_{n+1} = e^{-b([x_n]_q)^2} + c \quad \text{(q-Gaussian map)} \]  

(8)
or

\[ x_{n+1} = e^{-b \left( \frac{x_n}{1 - (1 - q)x_n - 1} \right)^2} + c \quad \text{(q-Gaussian map)} \]  

(9)
Under the limit \( q \to 1 \), the q-Gaussian map becomes the original Gaussian map (Eq. (7)). In this paper, throughout the discussion on \( q \)-Gaussian map, we prefer to use deformation parameter \( \epsilon \) instead of \( q \) (which are related by the relation \( 1 - q = \epsilon \)). Hence the explicit form of q-Gaussian map, which we use, is given as:

\[ x_{n+1} = e^{-\frac{\epsilon}{1 - \epsilon x_n}} + c, \quad \text{(q-Gaussian map)} \]  

(10)
which under the limit \( \epsilon \to 0 \), becomes the original Gaussian map (Eq. (7)).

2.2. The q-deformed Henon map

One of the simplest mathematical models, which exhibits strange attractor is the quadratic map introduced by Henon in 1976 [7]. The mathematical form of the Henon map is given by
If we generalize the above definition of q-deformation of 1D nonlinear map to the 2D case then we may introduce different deformation parameters for different state variables. Such generalized form of a 2D q-deformed map is given by

\[ x_{n+1} = f([x_n]_{q_x}, [y_n]_{q_y}) = 1 - \alpha x_n^2 + y_n \]  
\[ y_{n+1} = g([x_n]_{q_x}, [y_n]_{q_y}) = \beta x_n \]  
(2D q-deformed map)  
(12)

where \([x_n]_{q_x} = \frac{x_n}{1 - (1 - q_x)(x_n - 1)}\) and \([y_n]_{q_y} = \frac{y_n}{1 - (1 - q_y)(y_n - 1)}\).

Clearly under the limits \(q_x \to 1\) and \(q_y \to 1\) the q-deformed 2D map reduces to original map. The explicit form of the q-Henon map (by introducing \(1 - q_x = \epsilon_x\) and \(1 - q_y = \epsilon_y\), which we use throughout this study, is given by

\[ x_{n+1} = 1 - \alpha \left( \frac{x_n}{1 - \epsilon_x (x_n - 1)} \right)^2 + \left( \frac{y_n}{1 - \epsilon_y (y_n - 1)} \right) \]  
\[ y_{n+1} = \beta \left( \frac{x_n}{1 - \epsilon_x (x_n - 1)} \right) \]  
(13)

which under the limits \(\epsilon_x \to 0\) and \(\epsilon_y \to 0\), becomes the canonical Henon map (Eq. (11)). In the next section, we present results of our numerical exploration of the dynamical behaviour of above described q-deformed nonlinear maps.

### 3. Results and Discussion

A recent study [2] on q-deformation of nonlinear dynamical systems has revealed that the q-deformed logistic map exhibits a variety of dynamical behaviours: fixed point, periodic, chaotic and more interestingly the co-existence of attractors, which is a rare phenomenon in the 1D nonlinear maps. In the first part of our study we are interested in analyzing another 1D map-the Gaussian map, which is known to exhibit period doubling route to chaos and co-existing attractors [3,6] under the same q-deformation scheme. Recently, Patidar [6] has done a detailed study on the regions of the parameter space of Gaussian map where co-existing attractors exist. Here we embark to analyze the effect of q-deformation on the regions of the parameter space of Gaussian map where co-existing attractor exist. In Figure 1, we have shown the results of the analysis on the Gaussian map [3]. Particularly in Figure 1(a), we have identified different regions of the parameter space.
Figure 1. Parameter space \((b, c)\) of the Gaussian map (a) showing the regions, where chaotic (black shade) and regular (white shade) motions appear, (b) showing the regions where a period-1 attractor co-exists with some other periodic attractor (grey shade) and the regions where a period-1 attractor co-exists with a chaotic attractor (black shade). The results shown in Figure 1(a) are based on the extensive numerical calculation of Lyapunov exponent by iterating the Gaussian map 1000 times (after neglecting initial transient behaviour up to 400 iterations) at \(2 \times 10^6\) different points of the parameter space defined by \(0 \leq b \leq 20\) and \(-1.0 \leq c \leq 0\).
Chaotic situation has been recorded, whenever the Lyapunov exponent becomes greater than $1 \times 10^{-4}$. The calculation has been repeated for several sets of initial conditions to include all the co-existing attractors. In Figure 1(b), we have shown the complete region of parameter space where Gaussian map exhibits co-existing attractors.

Figure 2. Parameter space $(E, c)$ of the q-Gaussian map for $b=5.0$ (a) showing the regions, where chaotic (black shade) and regular (white shade) motions appear, (b) showing the regions where a period-1 attractor co-exists with some other periodic attractor (grey shade) and the regions where a period-1 attractor co-exists with a chaotic attractor (black shade).
Particularly, the grey shade shows the regions of the parameter, where a period-1 attractor co-exists with some other periodic attractor (i.e., period-1, period-2, period-4, ..., period-3, etc.), however the black shade shows the regions of the parameter space where a period-1 solution co-exists with a chaotic solution.

Figure 3. Parameter space (ε, c) of the q-Gaussian map for b=7.5 (a) showing the regions, where chaotic (black shade) and regular (white shade) motions appear, (b) showing the regions where a period-1 attractor co-exists with some other periodic attractor (grey shade) and the regions where a period-1 attractor co-exists with a chaotic attractor (black shade).
As explained in Section 2 that the q-deformation of any function/map is to introduce an additional parameter \( \epsilon \) in the definition of that function/map in such a way that under the limit \( \epsilon \to 0 \), the original function/map is recovered. The q-deformation of the Gaussian map (Eq. (10)) leads to a three parameter one dimensional nonlinear map. Now it becomes more complicated to analyze the dynamical behaviour of the q-Gaussian map in a three dimensional parameter space \((b, \epsilon, c)\). The effect of parameter \( b \) on the dynamical behaviour of Gaussian map has been analyzed [4], so we prefer to work in the two dimensional parameter space \((\epsilon, c)\) for some fixed values of parameter \( b \). The results of our analysis for the co-existing attractors in q-deformed Gaussian map (the analysis similar to the Gaussian map reported in Fig. 1) have been depicted in Figures 2 and 3 for \( b=5.0 \) and \( b=7.5 \) respectively. It can be easily observed from Figure 2 that for \( b=0.5 \), the non-deformed Gaussian map \((\epsilon = 0)\) there is a range of \( c \) for which co-existing attractors exist. For all the values of \( c \) which belong to this range, both the co-existent attractors are periodic i.e., chaotic solution does not co-exist with a period-1 attractor for any value of \( c \). Now if we deform the Gaussian map by changing the value of deformation parameter \( \epsilon \), then for all positive values of \( \epsilon \) both the co-existent attractors are periodic and the range, for which co-existing attractors exist, is decreasing with the increase in the value of \( \epsilon \) in positive direction. However, if we increase the value of deformation parameter \( \epsilon \) in the negative direction, initially the range of \( c \), for which co-existing attractors exist, increases and then after a particular value of \( \epsilon \), it starts decreasing. We also notice an important feature that for some negative values of deformation parameter \( \epsilon \), one of the co-existent attractors is chaotic. A similar feature we observe for \( b=7.5 \), the only difference is that the whole pattern is shifting towards the higher values of deformation parameters. In conclusion to the study on q-deformation of Gaussian map, we may infer that the q-deformation of the Gaussian does not lead to a drastic change in the dynamical behaviour (i.e., the qualitative behaviour is similar), however it introduce an additional parameter in the definition of the map, which sometime may be useful for making a choice of the desired dynamical behaviour required for some specific purposes (in case if we do not have direct access to change the parameters of the Gaussian map i.e. \( b \) and \( c \)). In the second part of our study, we analyze the dynamical behaviour of Henon map under the same q-deformation scheme. Since the Henon map is a 2D map and to consider the most general case, we introduce the two different deformation parameters corresponding to the deformation of two different state variables as explained in Section 2.2. Moreover in the canonical Henon map two system parameters are present hence after introducing the q-deformation, it becomes a four parameter system. To analyze the dynamical behaviour of this four parameter system, we choose fixed values for the parameter of canonical henon map i.e. \( \alpha \) and \( \beta \) and then analyze the the dynamical behaviour of the q-deformed system in the space of deformation
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Figure 4. Parameter space of the Henon map showing the regions correspond to variety of dynamical behaviours: fixed point solution (Black shade), periodic solutions (blue shade), chaotic solutions (red shade) and unbounded solutions (white shade).

Figure 5. Parameter space of the q-Henon map showing the regions correspond to variety of dynamical behaviours: fixed point solution (Black shade), periodic solutions (blue shade), chaotic solutions (red shade) and unbounded solutions (white shade).
parameters $\varepsilon_x$ and $\varepsilon_y$. In this way, we will be able to see the effect of q-deformation on the dynamical behaviour of Henon map for a particular choice of parameters $\alpha$ and $\beta$. In Figures 4 and 5, we have shown the results of one such analysis. Particularly in Figure 4, we have shown the regions of parameter space ($\alpha$, $\beta$) of non-deformed Henon map (i.e., $\varepsilon_x=\varepsilon_y=0$), where different different types of dynamics occur, i.e., fixed point solutions, periodic solutions, chaotic solutions and unbounded solutions (diverges to infinity), etc. These results are based on an extensive calculation of Lyapunov exponent at $15 \times 10^4$ different points of the parameter space defined by $-2 \leq \alpha \leq 3$ and $-2 \leq \beta \leq 1$. The black shade in Figure 4 represents the region of parameter space with fixed point solutions, blue and red shades represent periodic and chaotic solutions respectively and white shade represents the regions where dynamics diverges to infinity i.e., unbounded solutions. Now we choose a particular set of parameters $\alpha=1.4$ and $\beta=0.3$ for which a chaotic strange attractor exists and see the effect of q-deformation on the chaotic dynamics of the Henon map. The results of this analysis are shown in Figure 5, which is again based on extensive Lyapunov calculation at $4 \times 10^4$ points of the deformation parameter space defined by $-1 \leq \varepsilon_x \leq 1$ and $-1 \leq \beta \leq 1$. The same shading scheme has been used as in Figure 4. It is clear from Figure 5 that the q-deformation leads to the suppression of chaos in the Henon map as in the large part of the deformation parameter space the dynamics becomes periodic. Now several questions arise: (i) How this conversion from chaotic to periodic solution occurs in this case of q-deformation or in other words what’s the route to chaos in q-deformed Henon map?, Whether the topological shapes of the strange attractor of canonical Henon and q-deformed Henon maps are same or different?, Is it a universal feature in the all 2D q-deformed nonlinear map?, etc. Work in this direction is in progress and will be reported in the conference.

4. Conclusions
In this paper we investigated the dynamical behaviour of q-deformed Gaussian and Henon maps. We numerically explored the dynamics of these maps in the complete parameter space including the deformation parameters using extensive computation of the Lyapunov exponent. We observed a variety of dynamical behaviours and we are able to produce the desired behaviour by slightly changing the deformation parameter without disturbing the canonical system parameters of the system, which are not accessible in some practical situations. A more detailed analysis on the q-deformed Henon map is in progress to clarify the routes to the suppression of chaos due to the deformation and some universal features (if exist!) in such q-deformation of nonlinear maps. We strongly believe that such studies of q-deformation of
nonlinear maps can be used beneficially in modelling of several phenomena, which are not modeled exactly with the standard maps but their q-deformed versions could serve the purpose.

References


